1. Compute the value of the following integrals (*Hint: You might want to use Cauchy's integral formula to get the answer without long computations!*):

(a) $\displaystyle\int_\gamma \frac{e^{2z}}{z}\,dz$ with $\gamma = \{z : |z| = 2\}$ oriented counter-clockwise.

(b) $\displaystyle\int_\gamma \frac{z^3+2z^2+2}{z-2i}\,dz$ with $\gamma = \{z : |z-2i| = 1\}$ oriented clockwise.

(c) $\displaystyle\int_\gamma \frac{\sin(3z+\frac{\pi}{4})}{(z-\pi)^2}\,dz$ with $\gamma = \{z : |z-\pi| = 3\}$ oriented counter-clockwise.

2. Similarly, compute the following integrals:

(a) $\displaystyle\int_\gamma \frac{e^{2z}}{(z-1)(z^2+4)}\,dz$ where γ is the bounday of the rectangle

$$\mathcal{R} = \{z : -2 \leqslant \mathrm{Re}(z) \leqslant 2,\ -1 \leqslant \mathrm{Im}(z) \leqslant 1\}$$

oriented clockwise.

(b) $\displaystyle\int_\gamma \frac{\sin(z^2)}{\cos(z)}\,dz$ with $\gamma = \{z : |z-i| = 1\}$ oriented counter-clockwise. *(Hint: You might want to first compute the zeroes of the* $\cos(\cdot)$ *function by examining its real and imaginary parts.)*

3. Let γ be a *simple, closed and counter-clockwise oriented* regular curve in $\mathbb{C}$. What are the possible values the following integral can take (depending on the exact form of γ):

$$\int_\gamma \frac{\cosh(z^2+1)}{(z-2)^3}\,dz.$$

4. For

$$f(z) = \frac{e^{iz}}{(z-i)^2},$$

compute the integral $\displaystyle\int_\gamma f(z)\,dz$ in the following cases:

(a) $\gamma = \{z : |z| = 2\}$ oriented clockwise.

(b) γ is the boundary of the rectangle $\{z : |\mathrm{Re}(z)| \leqslant 4, |\mathrm{Im}(z)| \leqslant \frac{1}{2}\}$ oriented counter-clockwise.

(c) For some $R > 1$, γ is the closed curve formed by the union $\gamma_1 \cup \gamma_2$, where $\gamma_1(t) = t$ for $-R \leqslant t \leqslant R$ and $\gamma_2(s) = Re^{is}$ for $s \in [0, \pi]$ (draw a picture to visualise the curve in the comlex plane).
Bonus question: In the limit $R \to +\infty$, show that the integral $\int_{\gamma_2} f(z)\, dz$ goes to 0. Based on that, can you compute the value of $\int_{\mathbb{R}} f(z)\, dz$?

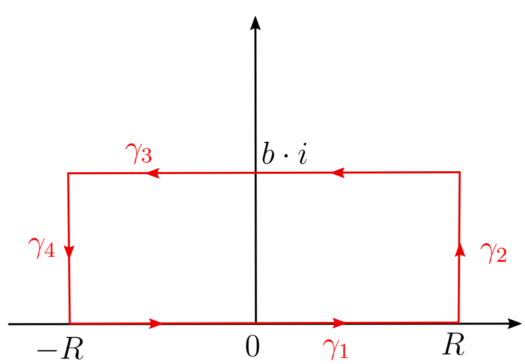
5. Complex integrals have historically proved to be a valuable tool in calculating complicated integral expressions, even in cases where the starting integrals do not seem to involve complex numbers at all. To illustrate this, we will use the techniques of complex integration to calculate

$$\int_{-\infty}^{+\infty} e^{-x^2} \cos(2bx)\, dx = \sqrt{\pi} e^{-b^2}, \tag{1}$$
$$\int_{-\infty}^{+\infty} e^{-x^2} \sin(2bx)\, dx = 0,$$

where $b > 0$ is any (real) given constant.

(a) Show that the function $f(z) = e^{-z^2}$ is entire.

(b) For any $R > 0$, consider the path $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ considered below:



Show that $\displaystyle\int_\gamma f(z)\, dz = 0$.

(c) Show that, as $R \to +\infty$, $\int_{\gamma_2} f(z)\, dz$, $\int_{\gamma_4} f(z)\, dz \to 0$.

(d) Using the fact that $\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}$, deduce from the above that (1) holds.

6. Let $\mathcal{D} \subset \mathbb{C}$ be an open set and $f : \mathcal{D} \to \mathbb{C}$ be a continuous function. Assume that there exists a holomorphic function $F : \mathcal{D} \to \mathbb{C}$ which is an antiderivative of f; this means that $F'(z) = f(z)$. Show that, for any regular curve $\gamma : [a, b] \to \mathcal{D}$,

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

(Hint: You will need to use the chain rule to compute $\frac{d}{dt}\big(F(\gamma(t))\big)$*.)*
In the case when $\mathcal{D} = \mathbb{C} \setminus \{0\}$ and $f(z) = \frac{1}{z}$, show that no antiderivative of f exists on $\mathcal{D}$. How is this consistent with what we know about $\log(z)$?

Remark: In the case when $\mathcal{D}$ is simply connected, the above formula can be used to show that any holomorphic $f : \mathcal{D} \to \mathbb{C}$ has an antiderivative.